P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Doomsday Equation The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the **doomsday equation.**

(a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Find the time T at which

$$\lim_{t \to T^-} y(t) = \infty.$$

(b) Solve the doomsday equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

given that $y(0) = y_0$. Explain why this equation is called the doomsday equation.

2. Sales Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

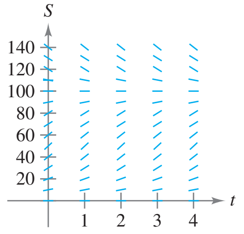
(a) Write the differential equation for the sales model when $L = 100$, $S = 10$ when $t = 0$, and $S = 20$ when $t = 1$. Verify that

$$S = \frac{L}{1 + Ce^{-kt}}.$$

(b) At what time is the growth in sales increasing most rapidly?

(c) Use a graphing utility to graph the sales function.

(d) Sketch the solution from part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(e) Assume the estimated maximum level of sales is correct. Use the slope field to describe the shape of the solution curves for sales when, at some period of time, sales exceed L.

3. Gompertz Equation Another model that can be used to represent population growth is the **Gompertz equation,** which is the solution of the differential equation

$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right) y$$

where k is a constant and L is the carrying capacity.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation when $k = 0.05$ and $L = 1000$.

(c) Describe the behavior of the graph as $t \to \infty$.

(d) Graph the equation you found in part (a) for $L = 5000$, $y_0 = 500$, and $k = 0.02$. Determine the concavity of the graph and how it compares with the general solution of the logistic differential equation.

4. Error Using Product Rule Although it is true for some functions f and g, a common mistake in calculus is to believe that the Product Rule for derivatives is $(fg)' = f'g'$.

(a) Given $g(x) = x$, find f such that $(fg)' = f'g'$.

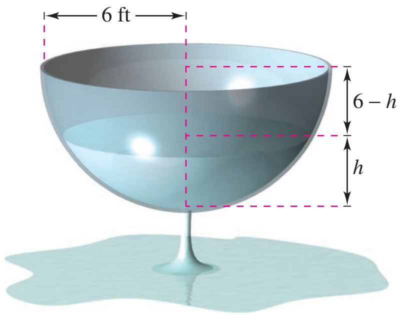
(b) Given an arbitrary function g, find a function f such that $(fg)' = f'g'$.

(c) Describe what happens if $g(x) = e^x$.

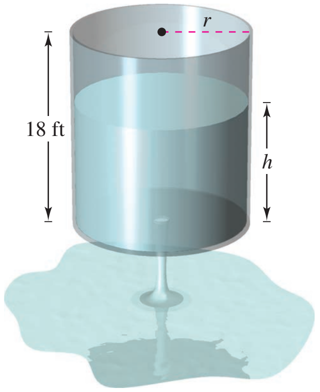
5. Torricelli's Law **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, $A(h)$ is the horizontal cross-sectional area at height h, and g is the acceleration due to gravity ($g \approx 32$ feet per second per second). A hemispherical water tank has a radius of 6 feet. When the tank is full, a circular valve with a radius of 1 inch is opened at the bottom, as shown in the figure. How long will it take for the tank to drain completely?



6. Torricelli's Law The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.



(a) Using Torricelli's Law, how long will it take for the tank to drain completely?

(b) What is the depth of the water in the tank after 1 hour?

7. Torricelli's Law Suppose the tank in Exercise 6 has a height of 20 feet and a radius of 8 feet, and the valve is circular with a radius of 2 inches. The tank is full when the valve is opened. How long will it take for the tank to drain completely?

8. Rewriting the Logistic Equation Show that the logistic equation

$$y = \frac{L}{1 + be^{-kt}}$$

can be written as

$$y = \frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right].$$

What can you conclude about the graph of the logistic equation?

9. Biomass Biomass is a measure of the amount of living matter in an ecosystem. Suppose the biomass $s(t)$ in a given ecosystem increases at a rate of about 3.5 tons per year, and decreases by about 1.9% per year. This situation can be modeled by the differential equation

$$\frac{ds}{dt} = 3.5 - 0.019s.$$

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation. What do you notice?

(c) Explain what happens as $t \to \infty$.

Medical Science In Exercises 10–12, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of the fluid in the compartment is constant.

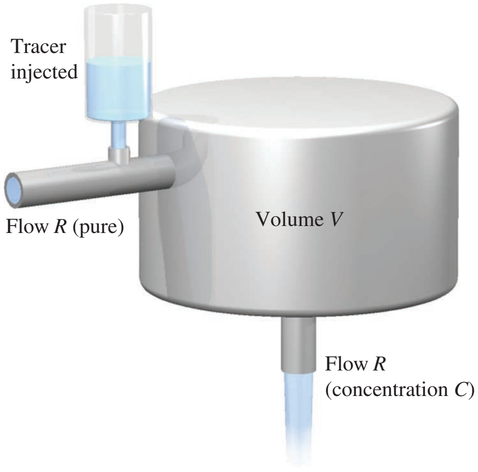


Figure for 10–12

10. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C$$

where $C = C_0$ when $t = 0$.

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.

11. Use the solution of the differential equation in Exercise 10 to find the concentration C as a function of time t, and use a graphing utility to graph the function.

(a) $V = 2$ liters, $R = 0.5$ liter per minute, and $C_0 = 0.6$ mole per liter

(b) $V = 2$ liters, $R = 1.5$ liters per minute, and $C_0 = 0.6$ mole per liter

12. In Exercises 10 and 11, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q moles per minute. Considering Q to be negligible compared with R, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C$$

where $C = 0$ when $t = 0$.

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.